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Full Additivity of the Entanglement of Formation

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We present a general strategy that allows a more flexible method for the construction of fully additive multipartite entanglement monotones than the ones so far reported in the literature of axiomatic entanglement measures. Within this framework we give a proof of a conjecture of outstanding implications in information theory: the full additivity of the Entanglement of Formation.

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Entangled states are the hallmark of quantum behaviour [2, 3, 4, 5]. A quantum system composed of two or more entangled subsystems has the interesting property that even though the state of the whole system can be well defined, it is impossible to assign individual properties to each of its parts [5, 6]. This fact has been at the heart of the conceptual foundations of quantum theory [2, 3, 4] and more recently has been re-discovered in terms of providing a very valuable tool as a physical resource for applications in quantum computation, information processing and communication [5, 6]. Within this context, it is imperative to have a procedure or scheme that allows a correct quantification of the degree of entanglement present in a given multipartite system at a given time. In this regard, several entanglement measures, from different perspectives, have been proposed in the past few years [7], but a final, generic, and ambiguity-free solution to this issue remains an open problem. In particular, within the axiomatic approach, there are several basic postulates that are essential to any construction of entanglement measures [7]. One central property of this approach is additivity, or even stronger, full additivity. In this paper we approach the problem of building general fully additive multipartite entanglement monotones. In so doing, we show how to simplify some of the conditions required for monotonicity, in particular that a function must be non-increasing under Local Operations and Classical Communications (LOCC).

The property of additivity is of particular interest to quantum information theory. We could translate it in mathematical and physical terms as follows. Suppose that two parties, Alice and Bob, share an EPR pair: they share a fixed value of entanglement. If, now, they are given another EPR pair, then their resources are doubled, as well as the entanglement they share. There are very few successful multipartite fully additive entanglement measures reported in the literature. In Ref. [1], a scheme based on total correlations measures and their mixed/pure convex roof constructions provided one such example. In this work we build a more flexible scenario for the construction of general fully additive multipartite entanglement monotones. In the second part of this work, we use this analysis to prove the, for a long time, conjectured additivity of the Entanglement of Formation (EoF) [9]. A lot of work has been devoted to proving this conjecture, since in Ref. [10] it was shown that i) additivity of the minimum entropy output of a quantum channel, ii) additivity of the Holevo capacity of a quantum channel, iii) additivity of the Entanglement of Formation and, iv) strong super additivity of the EoF, were equivalent problems. So far this has only been proven for some particular cases [11, 12]; however, a general proof of this has not yet been given.

In this paper we approach this problem from a different point of view and prove the conjecture. In doing so, we briefly recall some results reported in Ref. [1]. We then weaken these conditions and show how to obtain a more general scheme. Here, we derive a set of conditions (the α-conditions) which imply both LOCC monotonicity and full additivity. This procedure is conceptually depicted in Fig. 1. Using this, we proceed to apply the obtained theorem and thus prove the conjectured *full additivity* and *strong super additivity* of the bipartite EoF [8, 9]. We prove it by assuming weaker conditions for its multipartite generalization without the need to assume any special form for it.

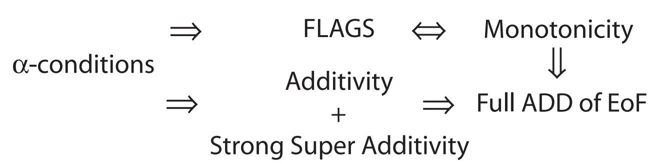

FIG. 1: Sketch of the general scheme followed in this work.

General strategy for building fully additive entanglement monotones.— For the purpose of the construction given in this Section, we first briefly recall some crucial results reported in Ref. [1], where the conditions for total correlations measures to be entanglement monotones were presented. We rely on two main results for building additive measures of entanglement [1]: 1) Let $\mathcal{T}$ be a measure of total correlations on pure states, and let $\mathcal{T}^*$ be its pure convex-roof extension. If $\mathcal{T}$ is a *complete total correlations measure* then $\mathcal{T}^*$ is a fully additive (ADD) and strongly super additive (SSA) quantity [14]. We say

it is also a *fully additive entanglement measure* if it is also an entanglement monotone. 2) Any complete total correlations measure $\mathcal{T}$, extended to mixed states through the pure convex-roof construction $\mathcal{T}^*$, is an entanglement monotone. These propositions have been demonstrated in Ref. [1]. This scenario, though general, can be specialized in order to get simpler and more flexible conditions for building fully additive multipartite entanglement monotones. The following theorem states one of the main findings reported in this paper.

Theorem .1 *Let $\mathcal{E}$ be a quantity defined from density matrices to real numbers, and let $\mathcal{E}^*$ be its pure convex-roof extension. $\mathcal{E}^*$ is a fully additive and strongly super additive entanglement monotone if it satisfies the following properties (α-conditions):*

FAEM1 Vanishing on separable pure states: $\mathcal{E}(|\Psi\rangle_1 \otimes ... \otimes |\Psi\rangle_N) = 0$.

FAEM2 Existence of maximally entangled states: There exist pure states $|\phi\rangle$ such that $\mathcal{E}(|\phi\rangle) \geq \mathcal{E}(|\Psi\rangle)$ for all $|\Psi\rangle$.

FAEM3 $\mathcal{E}$ is invariant under *local unitary* (LU) operations.

FAEM4 Strongly superadditive on pure states: $\mathcal{E}(|\Psi\rangle\langle\Psi|) \geq \mathcal{E}(\mathrm{Tr}_S[|\Psi\rangle\langle\Psi|]) + \mathcal{E}(\mathrm{Tr}_{\bar{S}}[|\Psi\rangle\langle\Psi|])$.

FAEM5 Additive on pure states: If $|\Psi\rangle = |\psi\rangle_{\bar{S}} \otimes |\psi\rangle_S$, then $\mathcal{E}(|\Psi\rangle) = \mathcal{E}(|\psi\rangle_{\bar{S}}) + \mathcal{E}(|\psi\rangle_S)$.

FAEM6 Pure convex-roof consistent:

$$\mathcal{E}(\rho) \geq \mathcal{E}^*(\rho) = \min \sum_i p_i \mathcal{E}(|\Psi^i\rangle\langle\Psi^i|). \qquad (1)$$

Note that if $\mathcal{E}$ is concave then Eq. (1) is automatically satisfied.

Proof i) *Full additivity and strong super additivity.* We initially consider the four-partite case, and then extend the argument to the N-partite system. Let's consider two two-qudit density matrices $\rho^{(1)}$ and $\rho^{(2)}$ with optimal decompositions $\rho^{(i)} = \sum p_a^{(i)} \sigma_a^{(i)}$, such that $\rho = \rho^{(1)} \otimes \rho^{(2)}$. We first consider an arbitrary non-bifactorizable decomposition and then show that this must have higher values for the convex-roof extension than those for the bifactorizable decomposition, thus showing that the latter decomposition is indeed the real minimum for the pure convex-roof construction:

$$\begin{aligned}
\mathcal{E}^*(\rho) &= \sum q_a \mathcal{E}\left(\rho_a^{1234}\right) && (2)\\
&\geq \sum q_a \left(\mathcal{E}\left(\rho_a^{12}\right) + \mathcal{E}\left(\rho_a^{34}\right)\right) && \text{(by FAEM4)}\\
&= \sum q_a \left(\mathcal{E}\left(\rho_a^{12}\right)\right) + \sum q_a \left(\mathcal{E}\left(\rho_a^{34}\right)\right)\\
&\geq \sum q_a \Big(\min \sum_s u_s^{(a)} \mathcal{E}\big(\rho_s^{(a)12}\big)\Big) + \text{i.d. over } \{34\}\\
&\geq \mathcal{T}^*\big(\rho^{(1)}\big) + \mathcal{T}^*\big(\rho^{(2)}\big)\ ,
\end{aligned}$$

where the last inequality follows as the decomposition resulting from minimizing every mixed density matrix in the expansion may not be the actual minimal decomposition of the complete matrix, i.e.,

$$\begin{aligned}
r_1\mathcal{E}\big(\eta^{(1)}\big) + r_2\mathcal{E}\big(\eta^{(2)}\big) &\geq \sum r_1 \Big(\min \sum_s u_s^{(1)} \mathcal{E}\big(\eta_s^{(1)}\big)\Big) + \\
&\quad \text{i.d. over } \{2\}\\
&\geq \mathcal{E}^*\left(\sum r_c \eta_c\right)\ . && (3)
\end{aligned}$$

For the N-partite case, we follow a similar line of reasoning as above. Here we consider two qudit registers, namely $\{U\}$ and $\{V\}$. Let u and v be the corresponding qudit density matrices $\rho^{(1)}$ and $\rho^{(2)}$ with optimal decompositions $\rho^{(i)} = \sum p_a^{(i)} \sigma_a^{(i)}$, such that $\rho = \rho^U \otimes \rho^V$. By the same token as before, we first consider an arbitrary non-bifactorizable decomposition, and then show that a bifactorizable decomposition is a lower bound for any possible non-bifactorizable decomposition:

$$\begin{aligned}
\mathcal{E}^*\left(\rho^{UV}\right) &= \sum q_a \mathcal{E}\left(\rho_a^{UV}\right) && (4)\\
&\geq \sum q_a \left(\mathcal{E}\left(\rho_a^U\right) + \mathcal{E}\left(\rho_a^V\right)\right) && \text{(by FAEM4)}\\
&= \sum q_a \left(\mathcal{E}\left(\rho_a^U\right)\right) + \sum q_a \left(\mathcal{E}\left(\rho_a^V\right)\right)\\
&\geq \sum q_a \mathcal{E}^*\left(\rho_a^U\right) + \sum q_a \mathcal{E}^*\left(\rho_a^V\right)\\
&\geq \mathcal{E}^*\left(\rho^U\right) + \mathcal{E}^*\left(\rho^V\right)\ ,
\end{aligned}$$

where the last inequality follows by the same argument that led to Eqs. (3).

Strong super additivity, i.e. $\mathcal{E}^*(\rho^{1,\dots,N}) \geq \mathcal{E}^*(\rho^{1,\dots,m}) + \mathcal{E}^*(\rho^{m+1,\dots,N})$, is demonstrated using the same reasoning as above, but with identifications $\rho = \rho^{1,\dots,N}$, $\rho_1 = \rho^{1,\dots,m}$, and $\rho_2 = \rho^{m+1,\dots,N}$. The last set of equations would then read

$$\mathcal{E}^*(\rho^{1,\dots,N}) \geq \mathcal{E}^*(\rho^{1,\dots,m}) + \mathcal{E}^*(\rho^{m+1,\dots,N})\ . \quad \blacksquare \qquad (5)$$

ii) *Monotonicity.* We only need to prove that a measure defined in this way is an LOCC non-increasing function, as the other properties are provided by the hypothesis. In so doing, we will make use of the FLAGS conditions introduced in Ref. [13]: an entanglement measure E is a monotone if and only if it is a local unitary invariant and satisfies

$$E\left(\sum p_i \rho_i \otimes |i\rangle\langle i|\right) = \sum p_i E(\rho_i)\ . \qquad (6)$$

To this end, we proceed in the following way. First, by convexity and FAEM5, we have

$$\mathcal{E}^*(\sum p_i \rho_i \otimes |i\rangle\langle i|) \leq \sum p_i \mathcal{E}^*(\rho_i \otimes |i\rangle\langle i|) = \sum p_i \mathcal{E}^*(\rho_i)\ .$$

Now we must show that $\mathcal{E}^*(\sum p_i \rho_i \otimes |i\rangle\langle i|) \geq \sum p_i \mathcal{E}^*(\rho_i)$ to get a full equality. To do this, we must show that the optimal decomposition of $\tilde{\rho} = \sum p_i \rho_i \otimes |i\rangle\langle i|$ is bounded

by $\sum p_i\mathcal{E}^*(\rho_i)$. Note that the above decomposition of $\tilde\rho$ implies that there exists a decomposition in pure states of the form

$$\rho = \sum_s \sum_i q_s p_i^{(s)} |\Psi_i^{(s)}\rangle\langle\Psi_i^{(s)}| \otimes |s\rangle\langle s| \ , \qquad (7)$$

which is valid as $\sum_{s,i} q_s p_i^{(s)} = 1$. We now show that if such a decomposition exists, then it minimizes $\mathcal{E}^*(\rho)$. As in previous cases, let's assume that the minimal decomposition is given by $\rho = t_a |\psi\rangle_a^{SR} \langle\psi|_a^{SR} = t_a \eta_a^{SR}$, where S may contain any number of qudits and R contains a single qudit. Then [14]

$$\begin{aligned}
\mathcal{E}^*(\rho) &= \sum t_a \mathcal{E}\left(\eta_a^{SR}\right) && (8)\\
&\geq \sum t_a \left(\mathcal{E}\left(\eta_a^{S}\right) + \mathcal{E}\left(\eta_a^{R}\right)\right) && \text{(by SSA)}\\
&\geq \sum t_a \left(\mathcal{E}^*\left(\eta_a^{S}\right) + \mathcal{E}^*\left(\eta_a^{R}\right)\right) && \text{(by PCRC)}\\
&= \sum t_a \left(\mathcal{E}^*\left(\eta_a^{S} \otimes \eta_a^{R}\right)\right) && \text{(by full ADD)}\\
&= \sum t_a \mathcal{E}^*\left(\eta_a^{S}\right) \ ,
\end{aligned}$$

where the last line follows as one qudit has no entanglement, i.e. $\mathcal{E}^*(\rho^{\dim(R)=1}) = 0$. This shows that for an arbitrary decomposition, $\rho^{SR} = p_i \rho_i^{SR}$, assumed to minimize $\mathcal{E}^*$, $\mathcal{E}^*(\rho_i^{SR}) \geq \sum t_a \mathcal{E}^*\left(\eta_a^S \otimes \eta_i^R\right) = \sum t_a \mathcal{E}^*(\eta_a^S)$, that is a decomposition of the form of Eq. (7), which exists by hypothesis, is a lower bound for it. However, we also showed in the previous step that for such a decomposition, $\mathcal{E}^*(\rho^{SR}) \leq \sum t_a \mathcal{E}^*\left(\eta_a^S \otimes \eta_a^R\right) = \sum t_a \mathcal{E}^*(\eta_a^S)$, thus

$$\mathcal{E}^*(\rho^{SR} = \sum t_a \eta_a^S \otimes \eta_a^R) = \sum t_a \mathcal{E}^*(\eta_a^S) \ , \qquad (9)$$

as claimed [15]. ∎

We are now ready to prove the following proposition.

Theorem .2 *Additivity of EoF.— Let E_F^b denote the bipartite Entanglement of Formation defined as*

$$E_F^b(\rho_{AB}) = \min \sum_i p_i S\left(\mathrm{Tr}_B \left|\Psi^i\right\rangle\left\langle\Psi^i\right|\right), \qquad (10)$$

and let E_F^m be its multipartite generalization

$$E_F^m(\rho) = \min \sum_i p_i E^m\left(\left|\Psi^i\right\rangle\left\langle\Psi^i\right|\right), \qquad (11)$$

where E^m is a multipartite pure state entanglement monotone, $E^m\big(|\Psi\rangle^{12}\big) = S\big(\mathrm{Tr}_2\big[\,|\Psi\rangle^{12}\,\big]\big)$.

If E^m exhibits strong superadditivity on pure states, and additivity on pure states, then E_F^b is fully additive.

Proof It follows from Theorem (.1) and the concavity of von Neumann's entropy which guarantees the PCRC condition. Note that we only require the multipartite generalization to exist and no explicit form of it is assumed.

To be explicit, we need to show that the bipartite Entanglement of Formation is additive, namely that if $\rho^{1234} = \rho^{12} \otimes \rho^{34}$, then $E_F^m(\rho^{1234}) = E_F^b(\rho^{12}) + E_F^b(\rho^{34})$. In so doing, we next show that given a bifactorizable density matrix $\rho^{1234} = \rho^{12} \otimes \rho^{34}$, with optimal decompositions $\rho^A = \sum p_i^{(A)} \rho_i^{(A)}$, the optimal decomposition is of the form $\rho^{1234} = \sum p_i^{(12)} \rho_i^{(12)} \otimes \sum p_i^{(34)} \rho_i^{(34)}$. Let's assume that the optimal decomposition of ρ^{1234} is not bifactorizable, i.e. is not of the form $\sum p_i^{(12)} \rho_i^{(12)} \otimes \sum p_i^{(34)} \rho_i^{(34)}$, then

$$\begin{aligned}
E_F^m(\rho^{1234}) &= \sum p_i^{(1234)} E^m\big(\rho_i^{(1234)}\big)\\
&\geq \sum p_i^{(1234)} \Big(E^m\big(\rho_i^{(12)}\big) + E^m\big(\rho_i^{(12)}\big)\Big)\\
&= \sum p_i^{(1234)} \Big(S\big(\mathrm{Tr}_2\big[\rho_i^{(12)}\big]\big) + S\big(\mathrm{Tr}_4\big[\rho_i^{(34)}\big]\big)\Big)\\
&\geq \sum p_i^{(1234)} \Big(E_F^m\big(\rho_i^{(12)}\big) + E_F^m\big(\rho_i^{(34)}\big)\Big)\\
&\geq E_F^b(\rho_{12}) + E_F^b(\rho_{34}) \ . && (12)
\end{aligned}$$

Thus, we have shown that given a non-bifactorizable decomposition of ρ^{1234}, it has a higher value of $\sum p_i^{(1234)} E^m\big(\rho_i^{(1234)}\big)$ than a bifactorizable decomposition, hence $E_F^m\big(\rho^{(12)} \otimes \rho^{(34)}\big) = E_F^b\big(\rho^{12}\big) + E_F^b\big(\rho^{34}\big)$. ∎

We note that this is a general proof of additivity of the EoF in the sense that it only requires its multipartite generalization to exist, without any assumption about its form. The conditions demanded are natural for any possible generalization. Furthermore, additivity and strong super additivity on pure states are natural conditions for a pure state entanglement monotone and are weaker than full additivity. Also, the bipartite EoF satisfies them trivially as one qubit has no entanglement. Note that we are not claiming the full additivity of E_F^m. It is clear that its full additivity would follow if it satisfies the PCRC condition.

There is, however, another scenario to consider. Let's suppose that we use the mixed convex-roof extension instead of the pure convex roof extension in the Theorem (.2). Following a similar reasoning to the one given above it is easy to prove that if Additivity and Strong super additivity of E^m on pure states are extended to [ADD] and [SSA] of E^m on general states, then E_F^m is a fully additive entanglement monotone as [PCRC (MCRC)] is granted by the mixed convex roof definition. Note that the measure of the bipartite Entanglement of Formation is not altered because of the concavity of von Neumann's entropy: in the bipartite case, both the mixed and pure convex roof extensions coincide, i.e. the minimum is attained over pure state decompositions.

We have proposed a strategy for quantifying entanglement in the multipartite case. The strategy generalizes the total correlations scheme introduced in Ref. [1] and allows a more flexible method for the construction of entanglement monotones. Furthermore, and independently

of the total correlations scheme, we have also stated and proved a general theorem for building fully additive entanglement monotones which simplifies the postulates of the axiomatic entanglement measures approach. Based on this theorem, we have given a proof of the long conjectured full additivity of the bipartite Entanglement of Formation. Additivity of the multipartite EoF in the mixed convex-roof scenario has also been addressed.

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[13] M. Horodecki, Open Syst. Inf. Dyn. **12**, 231 (2005).
[14] In the notation of Ref. [1], we recall that a *complete total* correlations function satisfies: i) *Additivity* [ADD]. Given two arbitrary states denoted by ρ_A and ρ_B, $\mathcal{T}(\rho_A \otimes \rho_B) = \mathcal{T}(\rho_A) + \mathcal{T}(\rho_B)$, ii) *Strong super additivity* [SSA]. Given a generic N-partite state $\rho^{1,\ldots,N}$, $\mathcal{T}(\rho^{1,\ldots,N}) \geq \mathcal{T}(\rho^{1,\ldots,m})+\mathcal{T}(\rho^{m+1,\ldots,N})$, iii) *Pure (mixed) convex-roof consistent* [PCRC (MCRC)]. This is the requirement that the convex-roof minimization is attained on decompositions over pure (mixed) states, which is equivalent to $\mathcal{T}(\rho) \geq \mathcal{T}^*(\rho) = \min \sum p_a \mathcal{T}(\rho_a)$, where the minimization is intended over pure (mixed) state decompositions of the state ρ.
[15] Note that this demostration is even more general as it doesn't require that $\eta_a^R = |a\rangle\langle a|$, only that it is a single qudit density matrix (FLAG).